Extremal and Probabilistic Graph Theory
Lecture 7
March 22nd, Tuesday

Recall. Moon-Moser inequality. If $N_{s-1} \neq 0$, then

$$\frac{N_{s+1}}{N_s} \geq \frac{s^2}{s-1}\left(\frac{N_s}{N_{s-1}} - \frac{n}{s^2}\right),$$

where N_s = number of K_s's in graph G.

We proved a stronger version:

Theorem (de-Daen). For r-graph G, if $N_{s-1} \neq 0$, then

$$\frac{N_{s+1}}{N_s} \geq \frac{s^2}{(s-r+1)(s+1)}\left(\frac{N_s}{N_{s-1}} - \frac{(r-1)(n-s)+s}{s^2}\right).$$

Corollary 7.1. *Suppose graph G has m edges s.t. $m = \left(1-\frac{1}{x}\right)\frac{n^2}{2}$ for some real $x > 0$. Then*

$$\frac{N_{s+1}}{N_s} \geq \frac{n(x-s)}{x(s+1)}.$$

Proof. We shall prove by induction on s. Base case ($s=1$) is clear, as $\frac{N_2}{N_1} = \frac{m}{n} \geq \frac{n(x-1)}{2x}$.

Assume it holds for s. Consider

$$\begin{aligned}\frac{N_{s+2}}{N_{s+1}} &\overset{M-M}{\geq} \frac{(s+1)^2}{s}\left(\frac{N_{s+1}}{N_s} - \frac{n}{(s+1)^2}\right)\\ &\overset{induction}{\geq} \frac{(s+1)^2}{s}\left(\frac{n(x-s)}{x(s+1)} - \frac{n}{(s+1)^2}\right)\\ &= \cdots = \frac{n(x-s-1)}{x(s+2)}\end{aligned}$$

∎

Corollary 7.2. *If $m > \left(1-\frac{1}{s}\right)\frac{n^2}{2}$, then $x > s$ and hence $N_{s+1}, N_s, ..., N_3 \geq 1$.*

Proof. For $\forall i \leq s$, $\frac{N_{i+1}}{N_i} > 0$. ∎

Remark. This implies the weaker version of Turán's Theorem.

Corollary 7.3. *If $m \geq \left(1-\frac{1}{s}+c\right)\frac{n^2}{2}$, then G has $\geq \frac{c}{2s^s}n^{s+1}$ copies of K_{s+1}.*

Proof. *Exercise* by corollary 7.1. ∎

We now consider the Hypergraph Turán Number for $K_s^{(r)}$. At the first lecture, we proved

$$\pi(K_s^{(r)}) \leq 1 - \frac{1}{\binom{s}{r}};\ \frac{5}{9} \leq \pi(K_4^3) \leq \frac{1}{\sqrt{2}}.$$

At present, it is not known the exact value of $\pi(K_s^{(r)})$ for any pair (s, r), where $s > r \geq 3$.

Theorem 7.4 (de-Caen)**.**

$$ex_r(n, K_k^{(r)}) \leq \left(1 - \frac{1}{\binom{k-1}{r-1}} + o(1)\right)\binom{n}{r}.$$

Remark. • It is the best upper bound for general $ex_r(n, K_k^{(r)})$.

- We will use the M-M inequality.

In fact, we will prove a slight strong statement.

Theorem 7.5 (Theorem')**.** *Let $k > r$. Let*

$$F(n, k, r) = \left[1 - \frac{n-k+1}{n-r+1}\frac{1}{\binom{k-1}{r-1}}\right]\binom{n}{r}.$$

Then

$$\frac{N_k}{N_{k-1}} \geq \frac{r^2\binom{k}{r}}{k^2\binom{n}{r-1}}(e(G) - F(n, k, r)).$$

Proof. We prove this by induction on k (fixed r).
Proof of base case $k = r + 1$:
Let $\mathcal{C}_s = \{$ all $K_s^{(r)}\}$.
For $e \in \mathcal{C}_s$, define $d(e) = \#\{f \in \mathcal{C}_{s+1} : e \subseteq f\}$.
For $e \in \binom{V}{r-1}$, let $d(e) = \{f \in \mathcal{C}_r : e \subseteq f\}$.
First,

$$\begin{aligned}
(r+1)N_{r+1} &= \sum_{e \in \mathcal{C}_r} d(e) \geq \sum_{e \in \mathcal{C}_r}\left(\sum_{e_i = e \setminus \{i\} \in \binom{V}{r-1}} d(e_i) - r - (r-1)(n-r)\right) \\
&= \sum_{f \in \binom{V}{r-1}} d^2(f) - m(r + (r-1)(n-r)) \\
&\geq \binom{n}{r-1}\left(\frac{\sum_{f \in \binom{V}{r-1}} d(f)}{\binom{n}{r-1}}\right)^2 - m(r + (r-1)(n-r)) \\
&= \frac{1}{\binom{n}{r-1}}(mr)^2 - m(r + (r-1)(n-r))
\end{aligned}$$

$$\Rightarrow \frac{N_{r+1}}{N_r} \geq \frac{r^2}{r+1}\frac{m}{\binom{n}{r-1}} - \frac{(r-1)(n-r)+r}{r+1}.$$

Exercise: this is identical to the inequality where $k = r+1$.

This proves the base case.

Now we suppose this theorem holds for k.

We show

$$\begin{aligned}
\frac{N_{k+1}}{N_k} &\overset{M-M}{\geq} \frac{k^2}{(k-r+1)(k+1)}\left[\frac{N_k}{N_{k-1}} - \frac{(r-1)(n-k)+k}{k^2}\right] \\
&\geq \frac{k^2}{(k-r+1)(k+1)}\left[\frac{r^2\binom{k}{r}}{k^2\binom{n}{r-1}}(m - F(n,k,r)) - \frac{(r-1)(n-k)+k}{k^2}\right] \\
&= \frac{r^2\binom{k}{r}}{(k-r+1)(k+1)\binom{n}{r-1}}\left[m - \binom{n}{r}\left(1 - \frac{n-k+1}{n-r+1}\frac{1}{\binom{k-1}{r-1}}\right)\right] - \frac{(r-1)(n-k)+k}{(k-r+1)(k+1)} \\
&= \frac{r^2\binom{k+1}{r}}{(k+1)^2\binom{n}{r-1}}\left[m - \binom{n}{r}\left(1 - \frac{n-k+1}{n-r+1}\frac{1}{\binom{k-1}{r-1}}\right)\right] - \frac{(r-1)(n-k)+k}{(k-r+1)(k+1)}
\end{aligned}$$

Need to show:

$$\begin{aligned}
&\frac{r^2\binom{k+1}{r}}{(k+1)^2\binom{n}{r-1}}\binom{n}{r}\left(1 - \frac{n-k+1}{n-r+1}\frac{1}{\binom{k-1}{r-1}}\right) + \frac{(r-1)(n-k)+k}{(k-r+1)(k+1)} \\
&\leq \frac{r^2\binom{k+1}{r}}{(k+1)^2\binom{n}{r-1}}F(n,k+1,r) \\
&= \frac{\binom{n}{r}}{\binom{n}{r-1}}\frac{r^2\binom{k+1}{r}}{(k+1)^2}\left(1 - \frac{n-k}{n-r+1}\frac{1}{\binom{k}{r-1}}\right).
\end{aligned}$$

In the end, they are equal! ∎

Proof of de-Caen's theorem (Using Theorem').

When $e(G) > F(n,k,r)$,

$$\frac{N_k}{N_{k-1}} \geq \frac{r^2\binom{k}{r}}{k^2\binom{n}{r-1}}(e(G) - F(n,k,r)) > 0$$

and

$$\frac{N_k}{N_{k-1}}, \cdots, \frac{N_{i+1}}{N_i} > 0$$

$$\Rightarrow N_k \geq 1, N_{k-1} \geq 1, ..., N_r \geq 1.$$

Thus

$$\begin{aligned}
ex_r(n, K_k^{(r)}) &\leq F(n,k,r) \\
&= \left[1 - \frac{n-k+1}{n-r+1}\frac{1}{\binom{k-1}{r-1}}\right]\binom{n}{r} \\
&\overset{k \geq r}{=} \left(1 - \frac{1}{\binom{k-1}{r-1}} + o(1)\right)\binom{n}{r}.
\end{aligned}$$

∎

We shall prove the following result in the coming lecture.

Theorem 7.6.

$$ex_r(n, K_k^{(r)}) \geq \left(1 - \left(\frac{r-1}{k-1}\right)^{r-1} - o(1)\right)\binom{n}{r}.$$

We mention two conjectures in this area.

Conjecture 7.7 (Turán)**.** $\pi(K_m^{(3)}) = 1 - \left(\frac{2}{m-1}\right)^2$.

Conjecture 7.8 (de-Caen)**.** $k\left[1 - \pi(K_{k+1}^{(k)})\right] \to +\infty$ *when* $k \to +\infty$.